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Bulk Viscous Bianchi Type-III Cosmological Model with Time-Dependent G and Λ

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Abstract This paper deals with Bianchi type-III anisotropic cosmological model of the universe filled with a bulk viscous fluid with time varying gravitational and cosmological constants. It is shown that the field equations are solvable for any arbitrary cosmic scale function. Exact solutions of Einstein's field equations are obtained which represent an expanding, shearing, non-rotating and decelerating universe. The physical behaviour of the model has also been discussed.

Keywords Bulk viscosity · Bianchi-III · Variable G and Λ

1 Introduction

One of the most important and outstanding problems in cosmology is the cosmological problem [1, 2]. A wide range of observations has suggested that the universe possesses a non-zero cosmological constant Λ which is considered as a measure of the energy-density of the vacuum [3]. The cosmological term provides a repulsive force opposing the gravitational pull between the galaxies. Linde [4] has suggested that Λ is a function of temperature and is related to the spontaneous symmetry breaking process, and therefore it could be a function of time. Recent discussions on the cosmological constant problem and consequence on cosmology with a time-varying cosmological constant are investigated by Ratra and Peebles [5], Dolgov [6–8] and Sahni and Starobinsky [9]. They have suggested that in the absence of any interaction with matter or radiation, the cosmological constant remains a constant. The existence of the cosmological term is favoured by recent supernovae (SNe) Ia observations [10–14] and which is also consistent with the recent anisotropy measurements of the cosmic microwave background (CMB) made by WMAP experiment [15].

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Variation of Newton's gravitational parameter G was originally suggested by Dirac on the basis of his large number hypothesis [16]. The Newtonian constant of gravitation G plays the role of coupling constant between geometry of the space-time and matter in Einstein's field equations. Therefore it is reasonable to consider G as time-dependent in an evolving universe when one considers time-dependent Λ. Many extensions of general relativity with time dependent G have been formulated ever since Dirac first considered the possibility of a variable G these theories has gained wide acceptance. A number of authors Sistero [17], Kalligas et al. [18], Arbab [19], Abdussattar and Vishwakarma [20] proposed a new approach which assumes the conservation of energy-momentum tensor which consequently renders G and Λ as coupled fields. This approach is physically appealing since it leaves the form of Einstein's equations formally unchanged by allowing a variation of G to be accompanied by change in Λ. Barrow and Parsons [21] have presented a detailed analysis of FRW universes in a wide range of scalar-tensor theories of gravity. Singh and Kotambkar [22] have discussed cosmological models with G and Λ in space-time of higher dimension. Vishwakarma [23] has investigated a Bianchi type I model with variable G and Λ, in which G, Λ and the shear parameter σ^2, all are coupled. The resulting model reduces to the FLRW model for large time with G approaching a constant. Singh and Sorokhaibam [24] have studied FRW cosmological models with the gravitational and cosmological constants. Singh et al. [25] have discussed Bianchi type-I cosmological models with variable G and Λ term in general relativity. Singh and Tiwari [26] have presented perfect fluid Bianchi type-I cosmological models with time-dependent G and Λ. Singh et al. [27] have obtained an exact solution of Einstein's field equations with variable G and Λ in the presence of perfect fluid for Bianchi type-III universe by assuming the conservation law for the energy momentum tensor. Chakraborty and Roy [28] have obtained anisotropic cosmological models with bulk viscous fluid in the presence of variable G and Λ. Singh and Kale [29] have presented Bianchi type-I, Kantowski-Sachs and Bianchi type-III anisotropic cosmological models of the universe filled with bulk viscous fluid together with variable G and Λ. Singh et al. [30], Pradhan and Kumhar [31] have studied anisotropic bulk viscous fluid cosmological models with varying Λ-term. Bali [32] has presented bulk viscous fluid cosmological models of Bianchi type III with time-dependent G and Λ by assuming the power forms of the scale factors.

Mazumdar [33] has shown that the field equations for LRS Bianchi type-I space-time with a perfect fluid are solvable for any arbitrary cosmic scale function and obtained exact solutions of the field equations in the systematic way. Recently, Katore and Rane [34] have investigated the inflationary Kantowski-sachs cosmological model in the presence of massless scalar field with a flat potential. In this paper, we have investigated Bianchi type-III cosmological model with varying cosmological and gravitational constants in presence of bulk viscous fluid in an expanding universe. The plan of the paper is as follows: The metric and the field equations are presented in Sect. 2. In Sect. 3, we deal with the solutions of the field equations. It is shown that the field equations are solvable for any arbitrary cosmic scale function and obtained exact solutions of the field equations. In Sect. 4, we have discussed the physical and geometric features of the expanding, shearing and non-rotating universe. We have given the concluding remarks in Sect. 5.

2 Metric and Field Equations

We consider the Bianchi type-III space-time represented by

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 e^{-2\alpha x} dy^2 + c^2 dz^2 \tag{1}$$

where A, B and C are function of cosmic time t alone and α is a constant.

Einstein's field equations with time-dependent G and Λ, in suitable units, are

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi G T_{ij} + \Lambda g_{ij}. \tag{2}$$

The energy momentum tensor for bulk viscous fluid is

$$T_{ij} = (\rho + \bar{p})\, v_i v_j + \bar{p} g_{ij} \tag{3}$$

where

$$\bar{p} = p - \xi v^i_{;i}. \tag{4}$$

Here ρ, p, $\bar{p}$ and ξ are respectively energy-density of matter, isotropic pressure, effective pressure and bulk viscosity coefficient and v_i, the four-velocity of the fluid satisfying.

$$v_i v^i = -1. \tag{5}$$

A semicolon stands for covariant differentiation. The coefficient of bulk viscosity determines the magnitude of the viscous stress relative to expansion.

An average expansion scale factor can be defined by $R(t) = (ABC)^{1/3}$ implying that the Hubble parameter $H = \frac{\dot{R}}{R}$. An overdot means ordinary differentiation with respect to cosmic time t. The physical quantities of dynamical interest in cosmology are the expansion scalar θ and the shear scalar σ. For the metric (1), the expressions for θ and σ are given by [35]:

$$\theta = v^i_{;i} = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{6}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right] - \frac{\theta^2}{6}. \tag{7}$$

An important observational quantity is the deceleration parameter q which is defined as

$$q = -\frac{R\ddot{R}}{\dot{R}^2}. \tag{8}$$

The sign of q indicates whether the model inflates or not. The positive sign corresponds to standard decelerating model whereas the negative sign indicates inflation.

Einstein's field equations (2) for the metric (1) in comoving coordinates lead to

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = -8\pi G\bar{p} + \Lambda, \tag{9}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} = -8\pi G\bar{p} + \Lambda, \tag{10}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\alpha^2}{A^2} = -8\pi G\bar{p} + \Lambda, \tag{11}$$

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{\alpha^2}{A^2} = 8\pi G\rho + \Lambda, \tag{12}$$

$$\alpha\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0. \tag{13}$$

An additional equation for time changes of G and Λ can be obtained by divergence of Einstein tensor which leads to $(8\pi G T_i^j - \Lambda g_i^j)_{;j}$, yielding

$$8\pi \dot{G}\rho + \dot{\Lambda} + 8\pi G\left[\dot{\rho} + (\rho + \bar{p})\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] = 0. \tag{14}$$

Using (4) the conservation of energy equation (14) splits in to two equations:

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0, \tag{15}$$

$$\dot{\Lambda} + 8\pi \dot{G}\rho = 8\pi G\xi\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)^2. \tag{16}$$

3 Solution of the Field Equations

Form (13), we have

$$\frac{\dot{A}}{A} = \frac{\dot{B}}{B} \tag{17}$$

since $\alpha \neq o$, which leads to

$$A = kB \tag{18}$$

where k is a constant of integration. Without loss of generality we can take $k = 1$.

From (9) and (11), we obtain

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \left(\frac{\dot{B}}{B}\right)^2 - \frac{\dot{B}}{B}\frac{\dot{C}}{C} = \left(\frac{\alpha}{B}\right)^2 \tag{19}$$

which on integration gives

$$-B^2\dot{C} + BC\dot{B} = \alpha^2\left[\int C dt + k_1\right] \tag{20}$$

where k_1 is an integration constant. We can write (20) in the form

$$\frac{d}{dt}(B^2) - 2\frac{\dot{C}}{C}(B^2) = F(t) \tag{21}$$

where

$$F(t) = \frac{2\alpha^2}{C}\left[\int C dt + k_1\right]. \tag{22}$$

The general solution of the linear differential equation (20) in B^2 is given by

$$B^2 = C^2\left[\int \frac{F(t)}{C^2} dt + k_2\right] \tag{23}$$

where k_2 is integration constant.

Note that in this case the solution of Einstein's field equation reduces to the integration of (23) if the explicit form of the scale factor C is known.

We obtain a particular solution of (23) for a simple choice of the function C. We choose

$$C = t^n \tag{24}$$

where n is a real number. Then (23) yields

$$B^2 = \frac{\alpha^2 t^2}{1-n^2} + \frac{2\alpha^2 k_1 t^{1-n}}{1-3n} + k_2 t^{2n}. \tag{25}$$

Without loss of generality, we take $k_1 = k_2 = 0$. Hence the solution (25) reduces to

$$B^2 = \frac{\alpha^2 t^2}{1-n^2}, \quad n \neq \pm 1. \tag{26}$$

The metric of the solution can be written in the form

$$ds^2 = -dt^2 + \frac{\alpha^2 t^2}{1-n^2}\left(dx^2 + e^{-2\alpha x} dy^2\right) + t^{2n} dz^2. \tag{27}$$

It is clear that, given $\xi(t)$, we can find the cosmological parameters. It most of the investigations involving bulk viscosity it is assumed to be a simple power function of the energy density [36, 37]:

$$\xi(t) = \xi_0 \rho^\beta \tag{28}$$

where ξ_0 and $\beta(> 0)$ are constant. For small density, β may even be equal to unity for simplicity [38]. The case $\beta = 1$ corresponds to a radiative fluid [37]. Near a big-bang, $0 \le \beta \le 1/2$ is a more appropriate assumption to obtain realistic models [39].

For the specification of ξ, we assume that the fluid obeys the equation of state

$$p = \gamma \rho \tag{29}$$

where γ $(0 \le \gamma \le 1)$ is a constant. Using (29) in (15), we obtain

$$\dot{\rho} + (\gamma + 1)\left(\frac{n+2}{t}\right)\rho = 0. \tag{30}$$

Integrating (30) yields

$$\rho = c_1 t^{-(\gamma+1)(n+2)}. \tag{31}$$

From (31) we have

$$\dot{\rho} = -\frac{c_1(\gamma+1)(n+2)}{t^{(\gamma+1)(n+2)+1}}. \tag{32}$$

Using the solution of A, B and C in (12), we obtain

$$\frac{n(n+2)}{t^2} = 8\pi G\rho + \Lambda, \tag{33}$$

which on differentiation leads to

$$\frac{-2n(n+2)}{t^3} = 8\pi \dot{G}\rho + 8\pi G\dot{\rho} + \dot{\Lambda}. \tag{34}$$

Substitution of (16) into (34) yields

$$8\pi G\dot{\rho} + 8\pi G\frac{(n+2)^2}{t^2}\xi = \frac{-2n(n+2)}{t^3}. \tag{35}$$

Again substituting (28) and (32) into (35), we obtain

$$G(t) = \frac{2n(n+1)}{t^3}\left[8\pi\left(\frac{c_1(\gamma+1)(n+2)}{t^{(\gamma+1)(n+2)+1}} - \frac{c_1^{\beta}\xi_0(n+2)^2}{t^{\beta(\gamma+1)(n+2)+2}}\right)\right]^{-1}. \tag{36}$$

From (31), (33) and (36), we find that

$$\Lambda(t) = \frac{n(n+2)}{t^2} - \left[\frac{2n(n+1)c_1}{t^{(\alpha+1)(n+2)2}}\right]\left[\frac{c_1(\gamma+1)(n+2)}{t^{(\gamma+1)(n+2)+1}} - \frac{c_1^{\beta}\xi_0(n+2)^2}{t^{\beta(\gamma+1)(n+2)+2}}\right]^{-1}. \tag{37}$$

The bulk viscosity coefficient has the value given by

$$\xi(t) = \xi_0 c_1^{\beta} t^{-\beta(\gamma+1)(n+2)}. \tag{38}$$

Thus, the bulk viscosity coefficient is a decreasing function of time.

4 Physical and Geometric Features

The model (27) represents an anisotropic Bianchi type-III bulk viscous fluid cosmological model with variable G and Λ. The physical and kinematical parameters of the model (27) are given by following expressions:
Spatial volume:

$$V^3 = \frac{\alpha^2}{1-n^2}t^{n+2}. \tag{39}$$

Expansion scalar:

$$\theta = \frac{(n+2)}{t}. \tag{40}$$

Shear scalar:

$$\sigma = \frac{(1-n)}{\sqrt{3}t}. \tag{41}$$

Deceleration parameter:

$$q = \frac{(1-n)}{(2+n)}. \tag{42}$$

In this model the value of deceleration parameter is positive which shows decelerating behaviour of the cosmological model. It is worthwhile to mention the work of Vishwakarma [40] where he has shown that the decelerating model is also consistent with recent CMB observations model by WNAP, as well as with the high redshift supernovae Ia data including 1997ff at $Z = 1.755$.

For the spatial volume to be positive, we must have $-1 < n < 1$. We observe that the spatial volume is zero at $t = 0$. Thus, the singularity exists at $t = 0$ in the model. The model starts evolving with a big-bang at $t = 0$ and the expansion decreases as time increases. At

$t = 0$, the physical parameters ρ, ξ, θ and σ are all infinite. The gravitational constant $G(t)$ is zero initially and gradually increases and tends to infinity at late times (i.e. $t \to \infty$). The cosmological term $\Lambda(t)$ is infinite at the beginning of the model and decreases as time increases and ultimately becomes zero at late time. As $t \to \infty$, the physical parameter ρ, ξ, θ and σ all tend to zero. Therefore the model essentially gives an empty space-time for large time. We also find that $\frac{\sigma}{\theta}$ tends to a constant limit as $t \to \infty$ which shows that the anisotropy in universe is maintained throughout.

5 Conclusion

Evolution of Bianchi type-III cosmological model is studied in the presence of a bulk viscous fluid with time-dependent G and Λ. The cosmic fluid satisfies the barotropic equation of state. We have shown that Einstein's field equations are solvable for any arbitrary cosmic scale function. All the matter and radiation are concentrated at the big-bang epoch $t = 0$. The model universe has a physical singularity at $t = 0$. The model represents an expanding, shearing and non-rotating universe.

We have shown regular well behaviour of energy density, bulk viscosity coefficient, gravitational constant $G(t)$, cosmological term $\Lambda(t)$ and the expansion of the universe with time. The cosmological term is found to be decreasing function of time and approaches to zero at late time which is supported by recent result from the observations of type Ia supernova explosion (SN Ia). The gravitational constant $G(t)$ is zero initially and gradually increases and tends to zero at late time. The role of bulk viscosity is also exhibited in this model of the decelerating universe. Since $\frac{\sigma}{\theta}$ tends to a constant as $t \to \infty$, the model does not approach isotropy and gives essentially an empty space for large time.

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